J. DIFFERENTIAL GEOMETRY
17 (1982) 239–253

ON THE CUSPIDAL SPECTRUM FOR FINITE VOLUME SYMMETRIC SPACES

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1. Introduction

Let $K \setminus G/\Gamma$ be a noncompact locally symmetric space of finite volume. Here G is a semisimple Lie group and Γ is an arithmetic subgroup. Moreover, K is a maximal compact subgroup.

If Δ is the Laplacian on $K \setminus G/\Gamma$, we consider Δ acting on the cuspidal functions $L^2_{\text{cusp}}(K \setminus G/\Gamma)$ in the sense of Langlands [14]. Our main result is the following:

Theorem 1.1. *Let $N(\lambda)$ be the number of linearly independent cuspidal eigenfunctions with eigenvalue less than λ. Then $N(\lambda)$ is finite for each fixed $\lambda > 0$.*

Moreover, one has the asymptotic upper bound:

$$\overline{\lim_{\lambda \to \infty}} \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2} \frac{\operatorname{vol}(K \setminus G/\Gamma)}{\Gamma(d/2+1)}. \tag{1.2}$$

Here d is the dimension of $K \setminus G/\Gamma$ and vol *denotes the volume. Also, $\Gamma(d/2+1)$ is the ordinary Gamma function.*

The fact that $N(\lambda)$ is finite for fixed $\lambda > 0$ was announced by Borel and Garland [2], [10].

If $G = \mathrm{SL}(2, R)$, then Theorem 1.1 has apparently been well known for some time. It certainly follows from the scattering theory of [15], although the explicit estimate is not stated there. Several authors [21] have given more detailed information for particular discrete subgroups Γ of $\mathrm{SL}(2, R)$. In the case $\Gamma = \mathrm{SL}(2, Z)$, equality holds in (1.2) and the limit on the left-hand side exists [15], [20].

When G is a real rank one, Gangolli and Warner [9] obtained the estimate $N(\lambda) \leqslant C\lambda^n$, for some C and n. However, their method did not give a good estimate of n.

Received November 13, 1981.

Theorem 1.1 was proved for real rank one in the author's earlier paper [6]. The arguments given below are a natural development of the approach initiated in this earlier work. Note that for the present paper, $K \setminus G/\Gamma$ may have arbitrary rank.

The author thanks the Australian National University for hospitality during the preparation of this manuscript.

2. Basic facts concerning arithmetic groups

This section summarizes some standard facts concerning semisimple Lie groups G and arithmetic subgroups Γ. For more details the reader is referred to [2] and [14].

Let $P = MAN$ be a parabolic subgroup of G. The parabolic subgroups P_θ belonging to P are in one-one correspondence with subsets θ of the simple roots Ψ of $\mathfrak{a}$, the Lie algebra of A. We may write $P_\theta = M_\theta A_\theta N_\theta$ where $N_\theta \subset N$, $A_\theta \subset A$, and $M_\theta \supset M$. The Lie algebra of N_θ consists of those positive roots containing at least one simple root not belonging to θ. We denote $S_\theta = M_\theta N_\theta$ and $S = MN$.

We denote the simple roots of $\mathfrak{a}$ by $\alpha_1, \alpha_2, \cdots, \alpha_k$. Set $A_c = \exp\{v \in \mathfrak{a} \mid \alpha_i(v) \geqslant c$, for all $i\}$. Here c is a real number and $\exp\colon \mathfrak{a} \to A$ is the diffeomorphism induced by the exponential map.

Suppose that P is a percuspidal parabolic in the sense of Langlands [14]. In particular, $\Gamma \cap P \subseteq S$ and $S/\Gamma \cap S$ is compact. Moreover, for any parabolic P_θ belonging to P one has $\Gamma \cap P_\theta \subseteq S_\theta$, $N_\theta/\Gamma \cap N_\theta$ is compact, and $S_\theta/\Gamma \cap S_\theta$ has finite volume. All percuspidal parabolics are conjugate in G.

If $P = MAN$ is any percuspidal parabolic, then set $\mathfrak{S}_c(P) = K \setminus MA_cN/\Gamma \cap P$, for any real number c. One may choose a finite set Ω of percuspidal subgroups P so that $K \setminus G/\Gamma$ is covered by $\bigcup_{P\in\Omega} \mathfrak{S}_c(P)$, for some real number c,

3. The metric on the cusp

Let $P = MAN$ be a percuspidal parabolic. The manifold with boundary $\mathfrak{S}_c(P)$ will be referred to as the cusp.

By proper choice of base point, we may assume that $K \cap P = K \cap M$, as is done in [4, p. 246]. We denote $K \setminus P_c = K \setminus MA_cN = ZA_cN$, where $Z = K \setminus M$. Then $K \setminus P_c$ is contained in $K \setminus G$ and the Killing form of G induces a right invariant metric on $K \setminus P_c$.

For each $(z, a) \in ZA_c$, the metric of $K \setminus P_c$ restricts to a metric on N. It is well known [4, p. 246] that this metric has uniformly bounded dependence on z, so the metric will be denoted by g_a. The crucial point is to understand the dependence of g_a upon a. One obtains a flat metric $\hat{g}_a$, on the Lie algebra $\mathfrak{n}$ of N, by identifying $\mathfrak{n}$ with the tangent space of N at the identity. Since N is a simply connected nilpotent Lie group, the exponential map exp: $\mathfrak{n} \to N$ is a diffeomorphism. Here we mean the group exponential map of N, which does not depend upon a choice of metric. Pulling back the metric $\hat{g}_a$ by $(\exp)^{-1}$ one may define a metric h_a on N.

It will be useful to employ a comparison of the metric g_a and h_a.

Lemma 3.1. *For ε sufficiently small, one has, in a g_a ball of radius ε about the identity element, $g_a \geqslant C_1 h_a$. Here C_1 is independent of a.*

Proof. For a fixed value a_0 of a one has, for some $\varepsilon > 0$, $g_{a_0} \geqslant C_1 h_{a_0}$, since exp is a diffeomorphism with differential the identity map. However, for any a, $zan = za_0(b^{-1}nb)b^{-1}$, where $b = a^{-1}a_0 \in A$. Since the Killing metric of $K \setminus P_c = ZA_cN$ is right invariant, $g_a = \mathrm{Ad}_b g_{a_0}$ and $\hat{g}_{a_0} = \mathrm{Ad}_b \hat{g}_{a_0}$. Notice that A normalizes N. The lemma now follows from the commutative diagram:

$$\begin{array}{ccc} \mathfrak{n} & \xrightarrow{\ \exp\ } & N \\ {\scriptstyle \mathrm{Ad}_b}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_b} \\ \mathfrak{n} & \xrightarrow{\ \exp\ } & N \end{array}$$

The metric $(d\omega)^2$ on $K \setminus P_c$ is described very explicitly in [4, p. 247]. In fact, one may write:

$$(d\omega)^2 = dz^2 + dr^2 + \sum_{\beta \in \Phi} e^{-2\beta(r)}\big(d\omega_\beta(z)\big)^2. \tag{3.2}$$

Here $r = (r_1, r_2, \cdots, r_k)$ are coordinates on A_c, obtained from the exponential map of A, exp: $\mathfrak{a} \to A$. In fact, $r_i(x) = \alpha_i(x)$, for $x \in \mathfrak{a}$, where α_i are the simple positive roots. Note that exp: $\mathfrak{a} \to A$ is a diffeomorphism, which allows us to identify $\mathfrak{a}$ with A. We may assume that A_c is parameterized by $r_i \geqslant c$, for all $1 \leqslant i \leqslant k$. The β belong to the set of positive roots Φ of $\mathfrak{a}$.

As given by (3.2), g_a is the right invariant metric on N which satisfies $g_a = \sum e^{-2\beta(r)}(d\omega_\beta(z))^2$ at the identity. It is difficult to obtain estimates on g_a directly since the distributions defined by the root spaces, i.e. the $d\omega_\beta(z)$ are not integrable. Thus g_a is not a product metric.

However, the metric h_a is a product metric, along the root spaces in $\mathfrak{n}$, which agrees with g_a at the identity. Of course, h_a is not right invariant with respect to N. Nevertheless, it is easier to estimate geometric quantities in h_a. This explains the utility of Lemma 3.1.

A key technical lemma is:

Lemma 3.3. *Let $\rho(x, y)$ denote the geodesic distance in the metric $(d\omega)^2$. Then one has, for ε sufficiently small, and any x, y, points in a fundamental domain for $\Gamma \cap N$:*

$$\sum_{\substack{\rho(x, y\gamma)<\varepsilon \\ 1\neq\gamma\in\Gamma\cap N}} \frac{1}{\rho(x, y\gamma)} \leqslant C_2\left(\max_{\substack{\alpha\in\Psi \\ n_\alpha=1}} \alpha(r)\right) \prod_{\beta\in\Phi} e^{n_\beta \beta(r)},$$

where n_β is the dimension of the root space corresponding to β. Here α runs over all simple positive roots of multiplicity one. The product in β runs over all positive roots. Moreover, $r = r(x)$, or if desired $r = r(y)$.

Proof. By Lemma 3.1 and formula (3.2), it suffices to obtain the analogous estimate for the Euclidean product metric h_a.

However, if ρ is the geodesic distance in h_a, one has

$$\sum_{\substack{\rho(x, y\gamma)<\varepsilon \\ 1\neq\gamma\in\Gamma\cap N}} \frac{1}{\rho(x, y\gamma)} \leqslant C_3 \sum_{\substack{\rho_\beta(x, y\gamma)\leqslant C_4 e^{\beta(r)} \\ \beta\in\Phi}} \left(\sum_\beta e^{-\beta(r)} \rho_\beta(x, y\gamma)\right)^{-1}, \tag{3.4}$$

where β are the positive roots of $\mathfrak{a}$ in $\mathfrak{n}$ and ρ_β is a fixed Euclidean metric on the root space corresponding to β. Thus ρ_β is independent of r.

A result of Moore [17, p. 155], states that the preimage of $\Gamma \cap N$ under exp: $\mathfrak{n} \to N$ is commensurable to a Euclidean lattice in the Lie algebra $\mathfrak{n}$. Using this fact, one obtains Lemma 3.3 after replacing the right sum in (3.4) by an integral:

$$\sum_{\substack{\rho(x, y\gamma)<\varepsilon \\ 1\neq\gamma\in\Gamma\cap N}} \frac{1}{\rho(x, y\gamma)} \leqslant C_5 \sum_{\alpha\in\Psi} \left(e^{\alpha(r)} \int_1^{C_4 e^{\alpha(r)}} t^{n_\alpha - 2}\, dt\right) \times \prod_{\beta\in\Phi-\alpha} \int_1^{C_4 e^{\beta(r)}} t^{n_\beta - 1}\, dt.$$

4. Neumann bracketing

Let $\phi \in L^2(K \setminus G/\Gamma)$ be a square integrable function. Suppose that P is a percuspidal parabolic and $P_\theta = M_\theta A_\theta N_\theta$ is any associated parabolic.

We may define

$$T(P, \theta)\phi(x) = \int_{N_\theta/\Gamma\cap N_\theta} \phi(xn)\, dn \tag{4.1}$$

for $x \in K \setminus G$. Here one has identified ϕ with a Γ invariant function on $K \setminus G$. Notice that the integral in (4.1) is well defined since $N_\theta/\Gamma \cap N_\theta$ is compact. If

$T(P, \theta)\phi = 0$ for all (P, θ), then ϕ is said to be cuspidal. If in addition $\Delta\phi = \mu\phi$, for some $\mu \geqslant 0$, then ϕ is a cuspidal eigenfunction and μ belongs to the cuspidal spectrum.

Choose a finite set $P_1, P_2, \cdots, P_r$ of percuspidal parabolics so that the collection $\mathfrak{S}_c(P_i)$, $1 \leqslant i \leqslant r$, covers $K \backslash G/\Gamma$. A function ψ on $\mathfrak{S}_c(P_i)$ is said to be cuspidal if $T(P_i, \theta)\psi = 0$, for the fixed parabolic P_i and all θ. Denote $\pi_i\colon \mathfrak{S}_c(P_i) \to K \backslash G/\Gamma$.

Now select a sequence of smooth compact manifolds with boundary $B_k \subset K \backslash G/\Gamma$ with $B_k \subset B_{k+1}$ and $UB_k = K \backslash G/\Gamma$. For each i and k, let $X_{i,k} \subset \mathfrak{S}_c(P_i)$ be a smooth manifold with boundary which contains $\mathfrak{S}_c(P_i) - \pi_i^{-1}B_k$. Suppose that $X_{i,k}N = X_{i,k}$, to guarantee that the cuspidal condition still makes sense in $L^2(X_{i,k})$. Eventually, we will wish to choose $X_{i,k}$ so that the volume of $X_{i,k}$ is sufficiently close to the volume of $\mathfrak{S}_c(P_i) - \pi_i^{-1}B_k$.

Let $\Delta_{i,k}$ be the Laplacian Δ acting on the cuspidal functions in $L^2(X_{i,k})$ which satisfy Neumann boundary conditions. Denote $N_{i,k}(\lambda)$ to be the number of cuspidal eigenfunctions in $L^2(X_{i,k})$ with eigenvalue less than λ. Similarly, we define $N_k(\lambda)$ to be the number of eigenvalues less than λ for the usual Neumann problem of the compact manifold with boundary B_k. It is not necessary to impose any cuspidal side condition in B_k.

The principle of modified Neumann bracketing developed in [6] and [15] now gives:

Proposition 4.2. *Let k be a fixed integer and suppose that $\Delta_{i,k}$ has pure point spectrum for all $1 \leqslant i \leqslant r$. If $N(\lambda)$ is the number of linearly independent cuspidal eigenfunctions on $K \backslash G/\Gamma$ with eigenvalue less than λ, then, for any value of λ:*

$$N(\lambda) \leqslant N_k(\lambda) + \sum_{i=1}^{r} N_{i,k}(\lambda).$$

A priori, $\Delta_{i,k}$ might have nonempty essential spectrum so that Proposition 4.1 would not apply. However, we will show presently that $\Delta_{i,k}$ does indeed have pure point spectrum for all i and k.

5. Interior parametrix

Let P be a fixed percuspidal parabolic. If $P_\theta = M_\theta A_\theta N_\theta$ is a cuspidal parabolic associated to P, then denote $T_\theta = T(P, \theta)$, where $T(P, \theta)$ is the cuspidal projection given by (4.1). We will normalize Haar measure on N_θ so that $\int_{N_\theta/\Gamma \cap N_\theta} dn_\theta = 1$. Recall that θ is a subset of the positive roots Ψ. It is convenient to set $\mathfrak{L}_\theta = T_{\Psi - \theta}$.

The following algebraic lemma is well known [11, p. 12]:

Lemma 5.1. (i) *For any* $\theta \subset \Psi$, *one has* $\mathfrak{L}_\theta = \prod_{\alpha \in \theta} \mathfrak{L}_\alpha$. *Here the product runs over simple positive roots contained in the subset* θ.

(ii) *For any* $\theta \subset \Psi$ *one has* $\mathfrak{L}_\theta^2 = \mathfrak{L}_\theta$.

(iii) *For any two subsets* $\theta_1, \theta_2 \subset \Psi$, *the associate projections commute*, $\mathfrak{L}_{\theta_1}\mathfrak{L}_{\theta_2} = \mathfrak{L}_{\theta_2}\mathfrak{L}_{\theta_1}$.

Now let $X \supset \mathfrak{S}_c(P) - \pi^{-1}B$ be a smooth manifold with boundary as chosen in §4. Recall that X depend upon integer parameters i, k. However, in the next two sections, both P and B are fixed so we will suppress the dependence upon i and k. Our eventual goal is to construct the fundamental solution of the heat equation problem with cuspidal interior conditions and Neumann boundary conditions on X. In this section, a parametrix satisfying the interior cuspidal conditions will be obtained. Lemma 5.1 is vital for this purpose.

Suppose $E(t, x, y)$ is the fundamental solution for the heat equation on the simply connected space $K \backslash G$. Then E is smooth on $(0, \infty) \times K \backslash G \times K \backslash G$ and satisfies the estimates [5]:

$$|E(t, x, y)| \leq C_1 t^{-d/2} \exp\left(\frac{-\rho^2(x, y)}{4t}\right),$$

$$\left|\frac{\partial E}{\partial \rho}(t, x, y)\right| \leq C_2 t^{-d/2}(\rho/t) \exp\left(\frac{-\rho^2(x, y)}{4t}\right) + C_3 t^{-d/2} \exp\left(\frac{-\rho^2(x, y)}{4t}\right) \tag{5.2}$$

uniformly for $0 < t \leq \tau$, any $\tau > 0$. Here $\rho(x, y)$ is the geodesic distance from x to y in $K \backslash G$ and d is the dimension of $K \backslash G$.

Let $P = MAN$. Then $G = KMAN$, and by proper choice of base point one has $K \backslash G = (K \cap M \backslash M)AN = ZAN$. Set $Y = K \backslash G / \Gamma \cap P = ZAN/\Gamma \cap P$. Then Y is a complete Riemannian manifold. Moreover, Y contains $\mathfrak{S}_c P = ZA_cN/\Gamma \cap P$, and therefore Y also contains the manifold with boundary X. In fact, X is an open set in Y.

Consider the infinite sum:

$$F(t, x, y) = \sum_{\gamma \in \Gamma \cap P} E(t, x, y\gamma). \tag{5.3}$$

By the results of [5], this sum converges uniformly on compact sets in $(0, \infty) \times K \backslash G \times K \backslash G$. Moreover, $F(t, x, y)$ represents the fundamental solution of the heat equation problem on Y.

Of course, $F(t, x, y)$ must be modified by projection onto the cuspidal conditions (4.1). Set

$$\bar{F}(t, x, y) = \prod_{\alpha \in \Psi} (1 - \mathcal{L}_\alpha(y)) F(t, x, y) \tag{5.4}$$
$$= \sum_{\theta \subset \Psi} (-1)^{|\theta|} \mathcal{L}_\theta(y) F(t, x, y).$$

Here the product runs over all simple roots and the sum runs over subsets θ of the simple roots. The projectors $\mathcal{L}_\theta(y)$ act on the third argument y of $F(t, x, y)$. It is immediate, from Lemma 5.1, that for all subsets $\psi \subset \Psi$, one has $\mathcal{L}_\psi(y)\bar{F}(t, x, y) = 0$. Thus F satisfies the cuspidal condition (4.1) and is suitable for an interior parametrix. By symmetry and isometry invariance of the heat kernel, one also has $\mathcal{L}_\psi(x)\bar{F}(t, x, y) = 0$, for all $\psi \subset \Psi$.

It is crucial to estimate the parametrix $\bar{F}(t, x, y)$ as a function of x and y for small $0 < t \leq \tau$, any fixed τ. For this purpose, we identify $x, y \in Y$ with points x, y in the universal cover $K \setminus G$, which realize the geodesic distance from x to y in Y.

Our basic technical estimate is:

Lemma 5.5. *For any fixed simple root* α, *let* $F_\alpha(t, x, y) = (1 - \mathcal{L}_\alpha(y))F(t, x, y)$. *Suppose that* $0 < t \leq \tau$, *where* τ *is fixed. One has the inequality*:

$$|F_\alpha(t, x, y)| \leq B_1 t^{-d/2} \min(e^{-\alpha r(x)}, e^{-\alpha r(y)}),$$
$$\times \max\left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))} \right)$$
$$\times \max_{\substack{\sigma \in \Psi \\ n_\sigma = 1}} \max_{x, y} (\sigma(r(x)), \sigma(r(y))) \exp(-\rho^2(x, y)/32t)$$

uniformly for $x, y \in \mathcal{S}_c(P)$, *any given* c. *Here* r *are the coordinates given by* (3.2) *and one uses the notation of Lemma* 3.3.

Proof. Let $\psi = \Psi - \alpha$ be the complement of α in Ψ. Then by definition:

$$F_\alpha(t, x, y) = \sum_{\gamma \in \Gamma \cap P} E(t, x, y\gamma) - \int_{N_\psi / N_\psi \cap \Gamma} E(t, xn, y\gamma)\, dn.$$

Using (5.2), we estimate the term coming from the identity element $\gamma = 1$:

$$F_\alpha(t, x, y) = \sum_{\substack{\gamma \in \Gamma \cap P \\ \gamma \neq 1}} E(t, x, y\gamma) - \int_{N_\psi / N_\psi \cap \Gamma} E(t, xn, y\gamma)$$
$$+ O\left(t^{-d/2} \exp(-\rho^2(x, y)/4t)\right).$$

The mean value theorem combined with (5.2) yields

$$|F_\alpha(t,x,y)| \leq B_2 t^{-d/2} \sum_{\substack{\gamma\in\Gamma\cap P\\ \gamma\neq 1}} \left[\frac{\rho(x,y\gamma)}{t} \exp\left(\frac{-\rho^2(x,y\gamma)}{8t}\right) + \exp\left(\frac{-\rho^2(x,y\gamma)}{8t}\right)\right] \times \min(\operatorname{diam}(x), \operatorname{diam}(y)) + O\left(t^{-d/2}\exp(-\rho^2(x,y)/4t)\right).$$

Here diam(x) is the diameter of $N_\psi/N_\psi \cap \Gamma$ at x. By formula (3.2), one has diam$(x) = O(e^{-\alpha(r)})$, where $r = r(x)$ are the coordinates on A_c used in (3.2).

It is an elementary lemma that we^{-w} is uniformly bounded for real $w \geq 0$. Consequently,

$$|F_\alpha(t,x,y)| \leq B_3 \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))}) t^{-d/2} \times \sum_{\substack{\gamma\in\Gamma\cap P\\ \gamma\neq 1}} \left[\frac{1}{\rho(x,y\gamma)} \exp\left(\frac{-\rho^2(x,y\gamma)}{16t}\right) + \exp\left(\frac{-\rho^2(x,y\gamma)}{8t}\right)\right] + O\left(t^{-d/2}\exp(-\rho^2(x,y)/4t)\right).$$

For any fixed $\varepsilon > 0$, we employ the estimate of [5, p. 491] to obtain:

$$|F_\alpha(t,x,y)| \leq B_4 t^{-d/2} \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))}) \times \left[\sum_{\substack{\gamma\in\Gamma\cap P\\ \gamma\neq 1\\ \rho(x,y\gamma)<\varepsilon}} \frac{1}{\rho(x,y\gamma)} \exp\left(\frac{-\rho^2(x,y\gamma)}{16t}\right) + \max(\mathrm{Vol}^{-1}(x), \mathrm{Vol}^{-1}(y)) \exp\left(\frac{-\rho^2(x,y)}{32t}\right)\right] + O\left(t^{-d/2}\exp(-\rho^2(x,y)/4t)\right).$$

Here $\mathrm{Vol}^{-1}(x) = 1/\mathrm{Vol}(x)$, and Vol($x$) is the volume of $N/\Gamma \cap N$ at x.

By formula (3.2), one has $\mathrm{Vol}^{-1}(x) = O(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))})$. Here Φ is the collection of positive roots of $\mathfrak{a}$ in $\mathfrak{n}$. Thus

$$|F_\alpha(t, x, y)| \leqslant B_5 t^{-d/2}\min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))})$$
$$\times\left[\sum_{\substack{\gamma\in\Gamma\cap P\\ \gamma\neq 1\\ \rho(x,y\gamma)<\varepsilon}} \frac{1}{\rho(x, y\gamma)}\exp\left(\frac{-\rho^2(x, y\gamma)}{16t}\right)\right.$$
$$\left.+\max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right)\exp\left(\frac{-\rho^2(x, y)}{32t}\right)\right].$$

If ε is sufficiently small, then referring to (3.2) we see that for $\gamma \in \Gamma \cap P$ and $\rho(x, y\gamma) < \varepsilon$, one must have $\gamma \in \Gamma \cap N$. Therefore Lemma 3.3 applies to yield:

$$|F_\alpha(t, x, y)| \leqslant B_1 t^{-d/2}\min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))})$$
$$\times\max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right)$$
$$\times \max_{\substack{\sigma\in\Psi\\ n_\sigma=1}}\max_{x,y}(\sigma(r(x)), \sigma(r(y)))\exp(-\rho^2(x, y)/32t).$$

Here σ runs over the simple roots of multiplicity one.

Using Lemma 5.5, it is easy to deduce:

Proposition 5.6. *If $\bar{F}(t, x, y)$ is the cuspidal parametrix defined by* (5.4), *then one has the estimate*:

$$|\bar{F}(t, x, y)| \leqslant B_7 t^{-d/2}\min_{\alpha\in\Psi}\min_{x,y}(e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}),$$
$$\times\max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right)\cdot\exp(-\rho^2(x, y)/32t)$$

uniformly for $x, y \in \mathfrak{S}_c(P)$, $0 < t \leqslant \tau$, for any given c and $\tau > 0$.

Proof. For any simple root α, we have

$$\bar{F}(t, x, y) = \prod_{\substack{\beta\in\Psi\\ \beta\neq\alpha}}(1 - \mathfrak{L}_\beta(y))F_\alpha(t, x, y).$$

Moreover, the projections $\mathfrak{L}_\beta(y)$, defined by (4.1), are L^∞-bounded.

Using the definition (4.1) and Lemma 5.5, one obtains immediately:

$$|\bar{F}(t,x,y)| \leqslant B_6 t^{-d/2} \min\left(e^{-\alpha(r(x))}, e^{-\alpha(r(y))}\right)$$
$$\times \max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right)$$
$$\times \max_{\substack{\sigma\in\Psi \\ n_\sigma=1}} \max_{x,y}\left(\sigma(r(x)), \sigma(r(y))\right) \exp\left(-\rho^2(x,y)/32t\right).$$

Here α is arbitrary.

Proposition 5.6 now follows by taking a minimum over α.

The same method gives estimates for the higher order derivatives of $F(t, x, y)$:

Proposition 5.7. *If $\bar{F}(t,x,y)$ is the cuspidal parametrix defined by* (5.4), *then one has the estimate*:

$$\left|\left(\frac{\partial}{\partial t}\right)^i \nabla_x^j \nabla_y^k \bar{F}(t,x,y)\right|$$
$$\leqslant B_8 t^{-d/2-i-j-k} \min_{\alpha\in\Psi} \min_{x,y}\left(e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}\right)$$
$$\times \max\left(\prod_{\beta\in\Phi} e^{n_\beta\beta(r(x))}, \prod_{\beta\in\Phi} e^{n_\beta\beta(r(y))}\right) \exp\left(-\rho^2(x,y)/32t\right).$$

Proof. First observe that the averaging process (4.1) commutes with covariant differentiation, since N acts isometrically. One then follows routinely through the proof of Proposition 5.6 and Lemma 5.5 by using the higher order estimates of [5]

6. Boundary corrections

Let W_1 be a complete Riemannian manifold without boundary containing a submanifold W_2 with boundary ∂W_2. We assume that W_1 and W_2 have the same dimension, i.e. the interior of W_1 is an open set in W_2. If W_1 and W_2 are compact, then given a fundamental solution F of the heat equation on W_1, the method of single layer potentials [19, pp. 175–194] allows one to modify F to obtain a fundamental solution of the heat equation on W_2 with Neumann boundary conditions. If the universal cover of W_1 has bounded geometry, i.e. the curvature is absolutely bounded and the injectivity radius is bounded below, and if ∂W_2 is compact, one can employ [5] to generalize the single layer potential construction given in [19]. However, when ∂W_2 is noncompact, further hypotheses are required.

We will use the single layer potential construction to modify $\bar{F}(t, x, y)$, given by (5.4), yielding a fundamental solution to the heat equation problem with Neumann boundary conditions on ∂X and cuspidal conditions on the interior of X. Here X is defined as in §5. Even though ∂X may be noncompact, its topology and geometry are precisely known outside a compact set. Thus, no serious difficulty arises when applying the methods of [19].

The basic estimates are the following:

Proposition 6.1. *Let $\bar{F}(t, x, y)$ be given by* (5.4). *Then set*

$$Q^{(0)}(t, x, y) = \bar{F}(t, x, y),$$

$$Q^{(m+1)}(t, x, y) = \int_0^t ds \int_{\partial X} \bar{F}(x, u, s) \frac{\partial}{\partial \nu} Q^{(m)}(u, y, t-s)\, du.$$

Here, the unit normal derivative $\partial/\partial\nu$ is applied to the argument u of $Q^{(m)}$.

One has the estimates, for $m \geqslant 1$:

$$\left|\left(\frac{\partial}{\partial t}\right)^i \nabla_x^j \nabla_y^k Q^{(m)}(t, x, y)\right| \leqslant C_1^m (\Gamma(m/2))^{-1} t^{-d/2-i-j-k}$$
$$\times \exp\left(-C_2\left(\sigma^2(x) + \sigma^2(y)\right)/t\right) \exp\left(-C_3 \rho^2(x, y)/t\right)$$
$$\times \min_{\alpha \in \Psi} \min_{x, y} \left(e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}\right)$$
$$\times \max\left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))} \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))}\right).$$

The notation is that of Proposition 5.7. *Moreover, $\sigma(x)$ is the distance from x to ∂X.*

Proof. The argument proceeds by induction starting from Proposition 5.7. One uses the method of [19] combined with the precise description of the metric on X given in (3.2). Since ∂X is given outside a compact set by $r_i = c$, for some i, in the coordinates of §3, the details are quite straightforward.

The fundamental solution is obtained as in [19].

Theorem 6.2. *Let $\bar{E}(t, x, y) = \sum_{m=0}^{\infty} (-2)^m Q^{(m)}(t, x, y)$, where $Q^{(m)}$ are given by Proposition* 6.1. *Then $\bar{E}$ is the fundamental solution of the heat equation with Neumann boundary conditions and cuspidal interior conditions on X.*

One has the estimate:

$$|\bar{E}(t, x, y)| \leqslant C_4 t^{-d/2} \min_{\alpha \in \Psi} \min_{x, y} \left(e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}\right)$$
$$\times \max\left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))}\right) \exp\left(\frac{-C_3 \rho^2(x, y)}{t}\right)$$
$$\times \left(1 + C_5 \exp\left(-C_2\left(\sigma^2(x) + \sigma^2(y)\right)/t\right)\right).$$

Here C_3 and C_4 depend only upon $Y = K\backslash G/\Gamma \cap P$. However, C_2 and C_5 are dependent upon the submanifold X.

Proof. From §5 and Proposition 6.1, it is clear that $\bar{E}(t, x, y)$ satisfies the heat equation and cuspidal interior conditions on X.

To show that $\bar{E}(t, x, y)$ satisfies Neumann boundary conditions one establishes the jump relations [19, p. 187] for the $Q^{(m)}$. This is primarily a local computation, which is undisturbed by the noncompactness of X.

The upper bound for $\bar{E}$ follows by writing $\bar{E} = Q^{(0)} + (\bar{E} - Q^{(0)})$ and quoting the estimates of Propositions 5.6 and 6.1.

7. Spectral function on the cusp

In this section we give an asymptotic upper bound for $N_X(\lambda)$. Here $N_X(\lambda)$ denotes the number of eigenvalues less than λ for the Laplacian with cuspidal interior conditions and Neumann boundary conditions on X, defined as in §6.

We begin with the following elementary lemma [7]:

Lemma 7.1. *Let B denote a nonnegative self adjoint operator acting on a Hilbert space. Suppose that the associated heat operator* $\exp(-tB)$ *is trace class, for all $t > 0$. Then B has pure point spectrum, so we may define $N_B(\lambda)$ as the number of eigenvalues of B less than λ. If, for some positive integer d,*

$$\overline{\lim_{t\to 0}}\, t^{d/2}\operatorname{Tr}(e^{-tB}) \leq D_1$$

then

$$\overline{\lim_{\lambda\to\infty}}\, \lambda^{-d/2} N_B(\lambda) \leq D_1 e.$$

Consider the Laplacian Δ_X acting on L^2X with Neumann boundary conditions and cuspidal interior conditions. The associated heat kernel $\bar{E}(t, x, y)$ for $\exp(-t\Delta_X)$ is estimated in Theorem 6.2. One may deduce:

Theorem 7.2. *The heat kernel $\bar{E}(t, x, y)$ defines a trace class operator* $\exp(-t\Delta_X)$. *Moreover, one has the estimate*:

$$\operatorname{Tr}(e^{-t\Delta_X}) \leq D_2 t^{-d/2} \int_{r(X)} \min_{\alpha\in\Psi} (e^{-\alpha(r)/2})\, dr + O(t^{-d/2+1/2}).$$

The constant D_2 depends only upon $Y = K\backslash G/\Gamma \cap P$. Here $r(X)$ is the set of r coordinates, as in (3.2), *for points in X.*

Proof. By the spectral theory of self adjoint operators, $\bar{E}(t, x, y)$ satisfies the semigroup property:

$$\bar{E}(t, x, y) = \int_X \bar{E}(t, x, z)\bar{E}(t, z, y)\, dz \tag{7.3}$$

and symmetry $\bar{E}(t, x, y) = \bar{E}(t, y, x)$.

Setting $x = y$, and integrating we find that

$$(7.4) \qquad \int_X |\bar{E}(t, x, y)|^2 dxdy = \int_X \bar{E}(t, x, x)\, dx.$$

The key estimate of Theorem 6.2 now gives, for small $t > 0$:

$$(7.5) \quad \int_X \bar{E}(t, x, x)\, dx \leq D_3 t^{-d/2} \int_{r(X)} \min_{\alpha \in \Psi} (e^{-\alpha(r)/2})\, dr + O(t^{-d/2+1/2}).$$

The integral on the right-hand side of (7.5) converges, so $\bar{E}$ is Hilbert-Schmidt by (7.4). However, the semigroup property (7.3) now shows that $\bar{E}$ is trace class. Then (7.5) gives the required upper bound for $\mathrm{Tr}(e^{-t\Delta_X})$.

It is convenient to denote $\mathfrak{M}(X) = \int_{r(X)} \min_{\alpha \in \Psi} (e^{-\alpha(r(x))/2})\, dr$.

From Lemma 7.1 and Theorem 7.2, one has immediately:

Corollary 7.6. *Let X be as in the first paragraph of this section. Then*

$$\overline{\lim_{\lambda \to \infty}}\, \lambda^{-d/2} N_X(\lambda) \leq D_4 \mathfrak{M}(X).$$

The constant D_4 depends only upon $Y = K \setminus G/\Gamma \cap P$. Otherwise, D_4 is independent of the particular choice of submanifold X.

8. Proof of the main theorem

It is now a straightforward matter to complete the proof of Theorem 1.1 of the introduction. Let B_k denote an exhaustion of $K \setminus G/\Gamma$ as in §4. Suppose $X_{i,k}$ are smooth manifolds in $\mathfrak{S}_c(P_i)$ as chosen there.

One has the asymptotic estimate of Minakshisundaram-Pleijel [1]:

Proposition 8.1. *Let W be a compact Riemannian manifold with boundary. If $N_W(\lambda)$ denotes the number of eigenvalues less than λ for the Neumann problem on W, then*

$$\lim_{\lambda \to \infty} \frac{N_W(\lambda)}{\lambda^{d/2}} = (4\pi)^{-d/2} \frac{\mathrm{Vol}(W)}{\Gamma(d/2+1)}.$$

Here d is the dimension of W and Vol(W) *is the volume of W.*

If $N(\lambda)$ is the number of cuspidal eigenvalues on $K \setminus G/\Gamma$ which are less than λ, then by Proposition 4.2, for any k:

$$\overline{\lim_{\lambda \to \infty}} \frac{N(\lambda)}{\lambda^{d/2}} \leq \overline{\lim_{\lambda \to \infty}} \frac{N_k(\lambda)}{\lambda^{d/2}} + \sum_{i=1}^{r} \overline{\lim_{\lambda \to \infty}} \frac{N_{i,k}(\lambda)}{\lambda^{d/2}}.$$

Here $N_k(\lambda)$ is the number of eigenvalues less than λ for the Neumann problem on the compact Riemannian manifold B_k.

Using Proposition 8.1 one obtains

$$\overline{\lim_{\lambda\to\infty}} \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2}\frac{\operatorname{Vol}(B_k)}{\Gamma(d/2+1)} + \sum_{i=1}^{r} \overline{\lim_{\lambda\to\infty}} \frac{N_{i,k}(\lambda)}{\lambda^{d/2}}.$$

Applying Corollary 7.6, one may deduce:

$$\overline{\lim_{\lambda\to\infty}} \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2}\frac{\operatorname{Vol}(B_k)}{\Gamma(d/2+1)} + \sum_{i=1}^{r} C_i \mathfrak{M}_i(X_{i,k}). \tag{8.2}$$

However, $\lim_{k\to\infty}\operatorname{Vol}(B_k) = \operatorname{Vol}(K\setminus G/\Gamma)$. Moreover, with a sensible choice of $X_{i,k}$, $\lim_{k\to\infty}\mathfrak{M}_i(X_{i,k}) = 0$, for all i.

Theorem 1.1 of the introduction follows by letting $k\to\infty$ in (8.2).

9. Coefficients in a bundle

The results derived above may be extended in a routine way to suitable differential operators acting on sections of equivariant vector bundles. In fact, the constructions of [5] are valid for any second order operator, which is G-invariant, and has leading symbol given by the metric tensor. Consequently, one may follow the previous sections of the present paper line by line to obtain:

Theorem 9.1. *Let ρ be any irreducible unitary representation of K, acting on a finite dimensional space of dimension* $\dim(\rho)$. *Suppose that $N(\lambda)$ is the number of cuspidal eigenfunctions less than λ for the Casimir operator acting on sections of the associated vector bundle $V_\rho\to G/K$. Then one has the asymptotic upper bound*:

$$\overline{\lim_{\lambda\to\infty}} \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2}\frac{\operatorname{Vol}(K\setminus G/\Gamma)}{\Gamma(d/2+1)}\dim(\rho).$$

By the argument of Matsushima-Murakami [16, p. 385], we may identify the Hodge Laplacian on p-forms with the Casimir operator on the bundle associated to the pth exterior power of the isotropy representation of K. Thus, a special case of Theorem 9.1 is:

Corollary 9.2. *Let $N(\lambda)$ be the number of cuspidal eigenfunctions with eigenvalue less than λ for the Hodge Laplacian acting on differential p-forms. Then one has the asymptotic upper bound*:

$$\overline{\lim} \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2}\frac{\operatorname{Vol}(K\setminus G/\Gamma)}{\Gamma(d/2+1)}\binom{d}{p}.$$

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